Road Capacity with a Steady Flow of Traffic

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Introductory: a problem that 1st or 2nd year students can analyze.

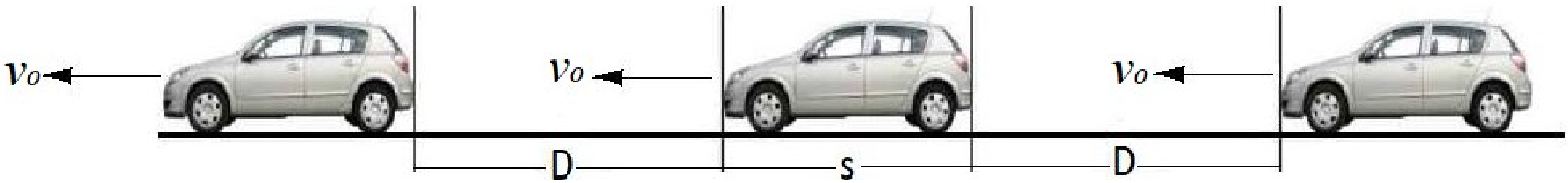
The new problem for introductory physics will be:

A simplified model of traffic flow optimization.

Flow: $\frac{\text{Number of cars}}{\text{unit of time}}$ passing a given observation point on a road.

Road Capacity: maximum possible safe flow.

The Setup



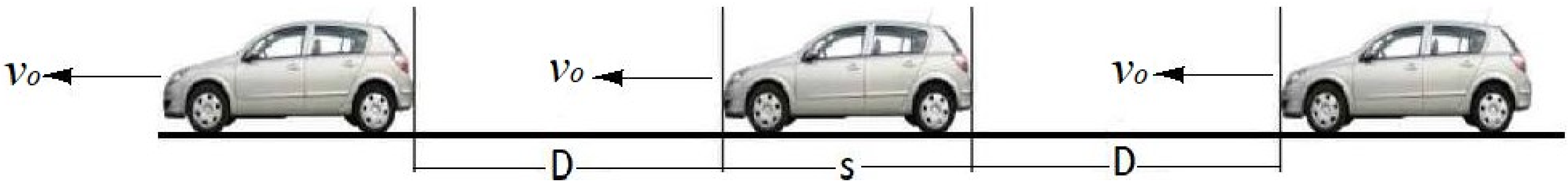
Single-lane road; identical cars at constant speed; equal spacing; and tire-road friction at $\boldsymbol{\mu}_s = 1$.

$v_{\mathbf{o}}$ (v-naught) because it is before possibly needed braking.

→What *speed* gets maximum safe flow?

→What is *your* intuitive estimate, based on experience?

(Calculation too involved to do in one's head.)



Flow: $\frac{\text{Number of cars}}{\text{unit of time}}$ passing a given observation point on a road.

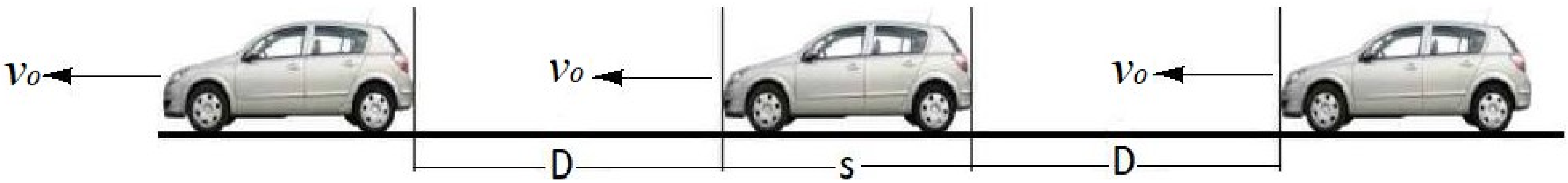
A fundamental equation for flow rate:

$\mathbf{j}=nv_o$ n ≡ linear density of cars (**not** just a "number" of cars**!**).

We use conventional hydrodynamic, electrostatic, and electric current notation.

From the picture:

$n=\frac{1}{s+D}$; combining: $\mathbf{j}=\frac{v_o}{s+D}$



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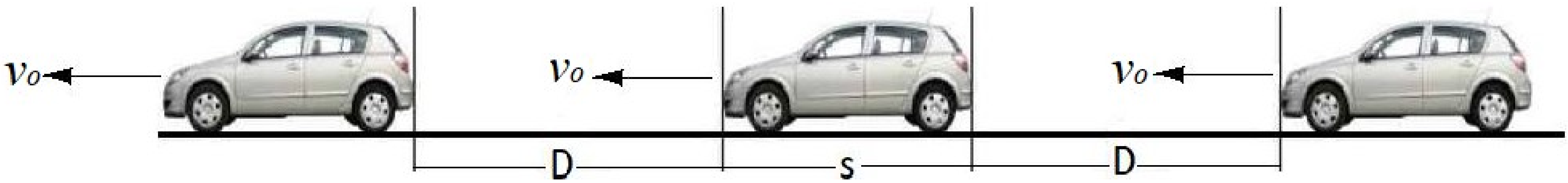
$j = nv_o$ since $n = \frac{1}{s+D}$, $j = \frac{v_o}{s+D}$

This is what we have so far.



Flow: $\frac{\text{Number of cars}}{\text{unit of time}}$ passing a given observation point on a road.

$\mathbf{j}=nv_o$ since $n=\frac{1}{s+D}$, $\mathbf{j}=\frac{v_o}{s+D}$.

How easy is it to significantly increase the flow rate ($\mathbf{j}$)?
Do we increase n? or do we increase v_o?

Larger n entails smaller D, which requires lower v_o for safety;
larger v_o requires larger D, which means smaller n.

If increase either n or v_o then the other one decreases, in $\mathbf{j}=nv_o$.

Clearly needed: {total stopping distance, **d**} ≤ {car spacing, D}.

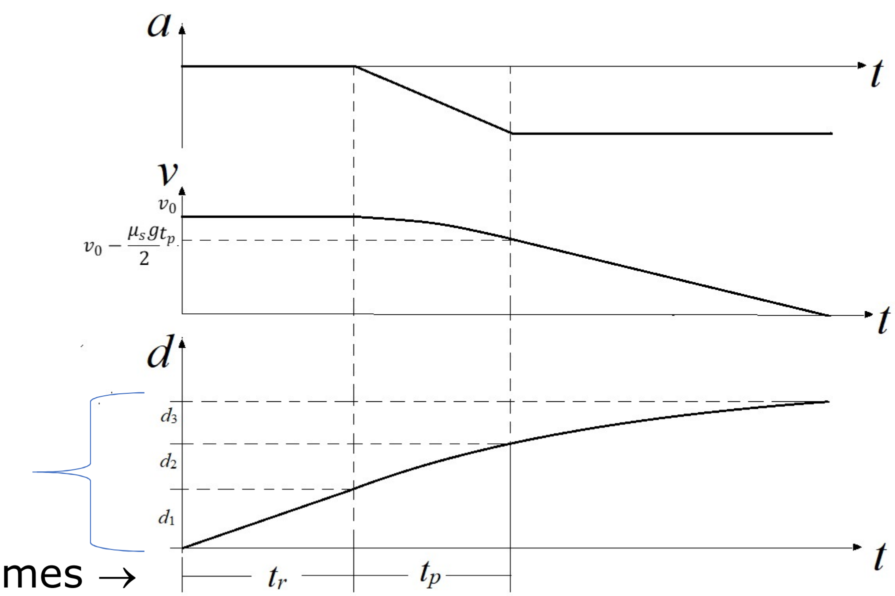
Distance traveled, because **while stopping,**

d	**there is time taken by**	**while acceleration is**
d_1	**thinking (reaction time)**	**zero**
d_2	**brake pressure buildup**	**linearly decreasing (becoming negative)**
d_3	**constant deceleration to v=0**	**constant<0**

d $= d_1 + d_2 + d_3$

The 3 d_i's →

and corresponding times →



Nice kinematic workout: find times and distances to stop.

Further stimulating student discussion, consider
 - cars' state of drivability
 - road condition
 - drivers' skills.
May need D significantly greater than just **d**
 i.e., (actual car spacing) > (theoretical stopping distance).

Expansion of D shown by: D = const·(stopping distance) = A·**d**.

Result: Replace D with actual necessary spacing of cars:
 - non-trivial formulas for stopping distance
 - margin for safety.

Earlier equation $j = \dfrac{v_o}{s+D}$ has become

$j = \dfrac{v_o}{s+A \cdot d}$. ←Note: “d”, not “D”.

Now, the theoretical d has turned this equation into:

$$j = \frac{v_o}{s + \{\text{polynomial with terms of } v_o \text{ and } v_o^2\}}$$

↪ {s, length of 1 car} > 0, so j has meaning when $v_o=0$

(the “parking lot case”)

$j(v_o=0)=0$ [due to numerator]

$j(v_o \rightarrow \infty)=0$ [due to v_o^2 in denominator]

⁂ j has a maximum for finite v_o.

Optimize $j = \dfrac{v_o}{s + \{\text{polynomial with terms of } v_o \text{ and } v_o^2\}}$ in order to

find $v_{o,opt}$ — and with it,

- ▣ $j(v_{o,opt}) \equiv j_{max} \equiv$ capacity (sometimes called **q**)
- ▣ $n_{opt} = j_{max}/v_{o,opt}$ = optimum car density
- ▣ $d(v_{o,opt})$ = total stopping distance.

Example:

let s = 5m = length of car

$A = 1.0$ as in normal intensive traffic

$t_{reaction} = 0.4$ s

$t_{pres} = 0.3$ s (time for brakes to build pressure),

then $v_{o,opt}$ = (... which you estimated earlier ...) 22 mph,

n_{opt} = 104 cars/mi, and

q = 0.64 s^{-1} = 1 car every 1.56 s = 2,300 cars/hr is the road capacity.

Experimental Values

Federal Highway Administration:
Revised Monograph on Traffic Theory, 2017
(chapter 2, Fig. 2.10)

their data: **q** = 2,200 cars/hr and $v_{o,opt}$ = 24 mph
this model: **q** = 2,300 cars/hr and $v_{o,opt}$ = 22 mph